Möbius Transformations For Global Intrinsic Symmetry Analysis

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Goal

- Find a map $f$ from surface onto itself that preserves geodesic distances

\[ f : \mathcal{M} \to \mathcal{M} \quad \text{s.t.} \quad d_g(p, q) = d_g(f(p), f(q)) \]
Previous Work

- Extrinsic Symmetry
- Intrinsic Symmetry
  - Symmetry Axis
  - Laplace-Beltrami Operator
  - Gromov-Hausdorff Distance
- Inter-Surface Correspondence
  - Möbius Voting

Podolak et al., 2006
Mitra et al., 2006
Previous Work

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Xu et al., 2009
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Ovsjanikov et al. '08
Previous Work

- Gromov-Hausdorff Distance
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Raviv et al. '10
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Previous Work
Möbius Voting

- Look for an isometry
  - Conformal
  - Area-preserving

- Conformal Maps
  - Mid-edge flattening
  - Möbius Transformation
  - Defined by 3 correspondences
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\[ m(z) = \frac{az + b}{cz + d} \]
Look for an isometry
- Conformal
- Area-preserving

Conformal Maps
- Mid-edge flattening
- Möbius Transformation
- Defined by 3 correspondences
Our Approach

- Look for an Anti-Möbius Transformation that makes intrinsic symmetry extrinsic on complex plane
Pipeline

Generating Set $S_1$

Complex Plane

Best Anti-Möbius Transformation

Final Correspondences

Correspondence Set $S_2$
Pipeline

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Final Correspondences
Finding a Symmetric Point Set

Generating Set $S_1$

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Final Correspondences
Finding a Symmetric Point Set

- Goal: need a set containing potential correspondences and stationary points
  
  e.g. Find a set $S \subset \mathcal{M}$ invariant under $f : f(S) = S$

- Approach: use critical points of symmetry invariant function $\Phi$
Finding a Symmetric Point Set

Example Symmetry Invariant Function

- Average Geodesic Distance
  \[ \Phi_{\text{agd}}(p) = \int_{d_g(p,q)} d_g(p,q) dq \]
Finding a Symmetric Point Set
Example Symmetry Invariant Function

- Average Geodesic Distance \( \Phi_{\text{agd}}(p) = \int_{d_g(p,q)} d_g(p, q) dq \)

- Robust to noise and outliers
- Only few extrema
- Generating Set for Anti-Möbius Transformations
Finding a Symmetric Point Set Theory

- Symmetry: $f : \mathcal{M} \rightarrow \mathcal{M}$ s.t. $d_g(p, q) = d_g(f(p), f(q))$
Finding a Symmetric Point Set

Theory

- **Symmetry:** $f : \mathcal{M} \to \mathcal{M}$ s.t. $d_g(p, q) = d_g(f(p), f(q))$
- **Symmetry Invariant Function:** $\Phi(p) = \Phi(f(p))$
Finding a Symmetric Point Set Theory

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- Prop. 7.1: \( \nabla|_p \Phi = 0 \iff \nabla|_{f(p)} \Phi = 0 \)
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Look for critical points
Finding a Symmetric Point Set
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- **Theorem 7.6**:  
  - If \( f \) is bilateral reflective, the gradient of \( \Phi \) is parallel to the curve of stationary points of \( f \)
Finding a Symmetric Point Set
Theory

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- **Theorem 7.6:**
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  At least 2 stationary points will have \( \nabla |_p \Phi = 0 \)
Finding a Symmetric Point Set

Theory

- **Symmetry:** $f : \mathcal{M} \rightarrow \mathcal{M}$ s.t. $d_g(p, q) = d_g(f(p), f(q))$
- **Symmetry Invariant Function:** $\Phi(p) = \Phi(f(p))$
- **Prop. 7.1:** $\nabla|_p \Phi = 0 \iff \nabla|_{f(p)} \Phi = 0$
  
  Look for critical points

- **Theorem 7.6:**
  - If $f$ is bilateral reflective, the gradient of $\Phi$ is parallel to the curve of stationary points of $f$
    At least 2 stationary points will have $\nabla|_p \Phi = 0$
  - For any other symmetry if there is a stationary point it would be a critical point of $\Phi$
Pipeline

Generating Set $S_1$

Complex Plane

Best Anti-Möbius Transformation

Final Correspondences

Correspondence Set $S_2$
Pipeline

Complex Plane

Best Anti-Möbius Transformation

Generating Set $S_1$

Correspondence Set $S_2$

Final Correspondences
Searching for the Best Anti-Möbius Transformation

- **Goal:** find a conformal map that is as isometric as possible
- **Approach:** use small symmetry invariant set to explore conformal mappings
Searching for the Best Anti-Möbius Transformation

- Explore all 3-plets:
  - \( z_1 \rightarrow z_1 \)
  - \( z_2 \rightarrow z_3 \)
  - \( z_3 \rightarrow z_2 \)

- Explore all 4-plets:
  - \( z_1 \rightarrow z_2 \)
  - \( z_2 \rightarrow z_1 \)
  - \( z_3 \rightarrow z_4 \)
  - \( z_4 \rightarrow z_3 \)

Symmetry Invariant Point Set from AGD (21 points)
Searching for the Best Anti-Möbius Transformation

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  - $z_4 \rightarrow z_3$

Symmetry Invariant Point Set from AGD (21 points)
Best Anti-Mobius Transformation

Green Edges: Mutually Closest Neighbors under a conformal map defined by $m$
Best Anti-Mobius Transformation

Alignment Score: How well does the map preserve area?

Good m
- 60% mutually closest

Bad m
- 17% mutually closest
Pruning

- Ignore a-priory bad mappings
  - Different AGD values
  - Too close correspondences
  - Different geodesic distances

Bad correspondence
Pruning

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  - Different AGD values
  - Too close correspondences
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Bad correspondence
Pruning

- Ignore a-priory bad mappings
  - Different AGD values
  - Too close correspondences
  - Different geodesic distances
Final Correspondences

- **Goal:** Given sparse correspondences: \((p_i, m(p_i) = q_i)\), find a correspondence \(q\) for every \(p\)
- **Approach:** For any \(p\), find \(q\) so that their geodesic distances to sparse set are same

Similar to: “Efficient computation of isometry-invariant distances between surfaces”. Bronstein et al. 2006
Pipeline

Generating Set $S_1$

Conformal Space

Best Anti-Möbius Transformation

Final Correspondences

Correspondence Set $S_2$
Results

Benchmark

- Goal: quantitatively evaluate performance of our method on 366 models

Scape: 71 Models

Non-Rigid World: 75 Models

SHREC, Watertight’07: 220 models
Results

Benchmark

- Ground Truth
- Geodesic Error
- Correspondence Rate
- Mesh Rate
- Results

\[ f_{\text{true}} : S_{\text{true}} \rightarrow S_{\text{true}} \]
Results

Benchmark

- Ground Truth
- Geodesic Error
- Correspondence Rate
- Mesh Rate
- Results

\[ f_{\text{true}} : S_{\text{true}} \rightarrow S_{\text{true}} \]
Results
Benchmark

- Ground Truth
- **Geodesic Error**
- Correspondence Rate
- Mesh Rate
- Results

\[ \sum_{s_{true} \in S_{true}} d_g(f(s_{true}), f_{true}(s_{true})) \]
Results

Benchmark

- Ground Truth
- Geodesic Error
- Correspondence Rate
- Mesh Rate
- Results

\[ d_g(f(s_{true}), f_{true}(s_{true})) < \tau \]
Results Benchmark

- Ground Truth
- Geodesic Error
- Correspondence Rate
- Mesh Rate
- Results

Correspondence Rate > 75%
Results
Benchmark

<table>
<thead>
<tr>
<th></th>
<th>Non-Rigid World</th>
<th>SCAPE Human</th>
<th>SHREC Watertight</th>
<th>All Data Sets</th>
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</thead>
<tbody>
<tr>
<td>Geodesic</td>
<td>3.3</td>
<td>4.2</td>
<td>1.93</td>
<td>2.65</td>
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<tr>
<td>Corr rate</td>
<td>85%</td>
<td>82%</td>
<td>83%</td>
<td>83%</td>
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<tr>
<td>Mesh rate</td>
<td>76%</td>
<td>72%</td>
<td>75%</td>
<td>75%</td>
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## Comparison

<table>
<thead>
<tr>
<th></th>
<th>Our Proposed Method</th>
<th>Mobius Voting (Lipman ‘09)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geodesic Corr rate (%)</td>
<td>3.49</td>
<td>6.78</td>
</tr>
<tr>
<td>Mesh rate (%)</td>
<td>86%</td>
<td>70%</td>
</tr>
<tr>
<td>Time (s)</td>
<td>25s</td>
<td>310s</td>
</tr>
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</table>
Rotational Symmetry
Large-scale outliers

Best Mobius

Second Best Mobius
Conclusion

- Anti-Mobius Transformations can be used for analysis of intrinsic symmetries
- Method succeeded on 75% of 366 meshes
- Our method improves speed and performance significantly over Möbius Voting
Limitations

- General partial intrinsic symmetries
  - Alignment error for a conformal map is global

- Symmetry-invariant sets
  - Robustness to noise
  - Various functions (other than AGD)
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  - Google
  - Rothschild Foundation

- **Data**
  - Daniela Giorgi and AIM@SHAPE (Watertight’07)
  - Drago Arguelov and Stanford University (SCAPE)
  - Project TOSCA (Non-Rigid World)
Online

• More data and results:

http://www.cs.princeton.edu/~vk/IntrinsicSymmetry/
Finding a Symmetric Point Set

- **Minimal Geodesic Distance**
  \[ \Phi_{\text{mgd}}(p; S_1) = \min_{q \in S_1} d_g(p, q) \]
Finding a Symmetric Point Set

- Minimal Geodesic Distance
  \[ \Phi_{mgd}(p; S_1) = \min_{q \in S_1} d_g(p, q) \]

- Can apply iteratively to construct set of arbitrary size
- Less robust
- Correspondence Set